A FROSTMAN TYPE LEMMA FOR SETS WITH LARGE INTERSECTIONS, AND AN APPLICATION TO DIOPHANTINE APPROXIMATION

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Abstract. We consider classes $\mathscr{G}^s([0,1])$ of subsets of $[0,1]$, originally introduced by Falconer, that are closed under countable intersections, and such that every set in the class has Hausdorff dimension at least s. We provide a Frostman type lemma to determine if a limsup-set is in such a class. Suppose $E = \limsup E_n \subset [0,1]$, and that μ_n are probability measures with support in E_n. If there is a constant C such that

$$\iint |x-y|^{-s} \, \mathrm{d}\mu_n(x)\mathrm{d}\mu_n(y) < C$$

for all n, then under suitable conditions on the limit measure of the sequence (μ_n), we prove that the set E is in the class $\mathscr{G}^s([0,1])$.

As an application we prove that for $\alpha > 1$ and almost all $\lambda \in (\frac{1}{2}, 1)$ the set

$$E_\lambda(\alpha) = \{\, x \in [0,1] : |x - s_n| < 2^{-\alpha n} \text{ infinitely often} \,\}$$

where $s_n \in \{\, (1-\lambda)\sum_{k=0}^{n} a_k \lambda^k \text{ and } a_k \in \{0,1\} \,\}$, belongs to the class $\mathscr{G}^s$ for $s \leq \frac{1}{\alpha}$. This improves one of our previous results in [5].

1. Introduction and Results

1.1. **Intersection classes.** Falconer introduced in [2] classes $\mathscr{G}^s$ of subsets of $\mathbb{R}^n$ with the property that any set in $\mathscr{G}^s$ has Hausdorff dimension at least s, and countable intersections of bi-Lipschitz images of sets from $\mathscr{G}^s$, are in $\mathscr{G}^s$. There are several equivalent ways to characterise the sets in $\mathscr{G}^s$, see [3]. We will use below a variant from Bugeaud [1]. (There is a minor mistake in the corresponding part in [3].)

We define the set functions $\mathscr{M}^t_\infty$ on arbitrary sets $E \subset \mathbb{R}^n$ as

$$\mathscr{M}^t_\infty(E) = \inf \Bigl\{ \sum_i |D_i|^t : E \subset \bigcup_i D_i \Bigr\},$$

where each D_i is a dyadic hypercube. According to [1], $\mathscr{G}^s$ is the class of G_δ sets E such that for each $t < s$, there is a constant c such that

$$\mathscr{M}^t_\infty(E \cap D) \geq c \mathscr{M}^t_\infty(D) \tag{1}$$

holds for all dyadic cubes D.

If E_n are open sets and $E = \limsup E_n$, then (1) holds and E is in $\mathscr{G}^s$, provided that

$$\liminf_{n\to\infty} \mathscr{M}^t_\infty(E_n \cap D) \geq c|D|^t \tag{2}$$

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holds for all dyadic cubes D, see [3]. We will use a small variation of this result, as stated in Section 2.1.

In this paper, we will consider subsets of the interval $[0,1]$. Since no subset of $[0,1]$ belongs to the class $\mathscr{G}^s$, we introduce instead the class $\mathscr{G}^s([0,1])$, which is the analog of $\mathscr{G}^s$ for subsets of $[0,1]$. The class $\mathscr{G}^s([0,1])$ is the class of G_δ subsets E of $[0,1]$ such that if we deploy copies of E, translated by an integer, along the real line, then we get a set that belongs to the class $\mathscr{G}^s$. Equivalently, $\mathscr{G}^s([0,1])$ can be defined as $\mathscr{G}^s$, using (1), where we instead require that (1) holds for all dyadic cubes D that are subsets of $[0,1]$. With the same change, (2) can be used to determine if a set belongs to the class $\mathscr{G}^s([0,1])$.

Our first result is the following theorem, that gives a method to determine if a limsup-set belongs to the class $\mathscr{G}^s([0,1])$.

Theorem 1. *Let E_k be open subsets of $[0,1]$, and μ_k Borel probability measures, with support in the closure of E_k, that converge weakly to a measure μ with density h in L^2. Assume that $\mu(I) > 0$ for all intervals $I \subset [0,1]$ with non-empty interior, and assume that for each $\varepsilon > 0$, there is a constant C_ε, such that*

$$|I|^{1+\varepsilon} \|h\chi_I\|_2^2 \leq C_\varepsilon \|h\chi_I\|_1^2 \tag{3}$$

holds for any interval $I \subset [0,1]$. If there is a constant C such that

$$\iint |x-y|^{-s} \, \mathrm{d}\mu_k(x) \mathrm{d}\mu_k(y) \leq C \tag{4}$$

holds for all k, then $\limsup E_k$ is in the class $\mathscr{G}^s([0,1])$.

We will prove Theorem 1 in Section 2. Next, we will present our application of this theorem.

1.2. **Diophantine approximation with $\boldsymbol{\lambda}$-expansions.** Let $\lambda \in (\frac{1}{2}, 1)$, and $\alpha > 1$. We consider the sets

$$E_\lambda(\alpha) = \{\, x \in [0,1] : |x - s_n| < 2^{-\alpha n} \text{ for some } s_n \text{ infinitely often} \,\}$$

where $s_n \in \{\, (1-\lambda)\sum_{k=0}^n a_k \lambda^k \text{ and } a_k \in \{0,1\} \,\}$. This set can be written as a limsup-set, $E_\lambda(\alpha) = \limsup_{n\to\infty} E_{\lambda,n}(\alpha)$, where

$$E_{\lambda,n}(\alpha) = \{\, x \in [0,1] : |x-y| < 2^{-\alpha n} \text{ for some } y \in F_{\lambda,n} \,\},$$

$$F_{\lambda,n} = \{\, y : y = (1-\lambda)\sum_{k=0}^n a_k \lambda^k,\ a_k \in \{0,1\} \,\}.$$

The membership in the classes $\mathscr{G}^s([0,1])$ of the set $E_\lambda(\alpha)$ for typical λ, was studied in our paper [5], where it was proved that $E_\lambda(\alpha)$ belongs to $\mathscr{G}^{1/\alpha}([0,1])$ for almost all $\lambda \in (\frac{1}{2}, \frac{2}{3})$. Here, we can state the following improvement of this result.

Theorem 2. *For almost all $\lambda \in (\frac{1}{2}, 1)$, the set $E_\lambda(\alpha)$ is in $\mathscr{G}^{1/\alpha}([0,1])$.*

Remark 1. We note that we cannot have $E_\lambda(\alpha) \in \mathscr{G}^s([0,1])$ for any $s > 1/\alpha$, since a simple covering argument shows that the Hausdorff dimension of $E_\lambda(\alpha)$ is not larger than $1/\alpha$.

We should also remark that in our paper [5], we studied a different scaling of the sets $E_\lambda(\alpha)$, so that they had diameter $\lambda/(1-\lambda)$. This is unimportant for the result. In this paper it will prove more convenient to work with the sets $E_\lambda(\alpha)$ if they are all subsets of $[0,1]$, hence the difference.

The proof of Theorem 2 is in Section 3. It is an application of Theorem 1.

2. Proof of Theorem 1

2.1. **A Lemma on Large Intersection Classes.** We start with the following lemma, that will be used later in the proof. It is the previously mentioned variation of (2).

Lemma 1. *Let E_n be open sets and $E = \limsup E_n$. If for any $\varepsilon > 0$ and $t < s$ there is a constant $c_{t,\varepsilon}$ such that*

$$\liminf_{n\to\infty} \mathscr{M}^t_\infty(E_n \cap D) \geq c_{t,\varepsilon}|D|^{t+\varepsilon}$$

holds for all dyadic cubes $D \subset [0,1]$, then E is in the class $\mathscr{G}^s([0,1])$.

The proof is a minor perturbation of the proof of Lemma 2 in [3].

Proof. Let $0 < t < u < s$ and $\varepsilon > 0$. We take a dyadic cube D of length 2^{-m}, and choose a number $n \geq m$ such that

$$2^{-n(t-u)} \geq c_{u,\varepsilon}^{-1} 2^{-m(t-u-\varepsilon)}.$$

Let $\{I_i\}$ be any disjoint cover of $E \cap D$ by dyadic cubes. We write D as a finite union of disjoint dyadic cubes,

$$D = \bigcup_{j=1}^{k} J_j,$$

such that for any j either

i) $J_j = I_i$ for some i and $|J_j| > 2^{-n}$,

or

ii) $|J_j| = 2^{-n}$ and those I_i that cover $E \cap J_j$ are subsets of J_j.

Let $Q(j) = \{\, i : I_i \subseteq J_j \,\}$. If j satisfies i), then $Q(j)$ has exactly one element, and so

$$\sum_{i\in Q(j)} |I_i|^t = |J_j|^t = |J_j|^{t-u}|J_j|^u \geq |D|^{t-u}|J_j|^u \geq |D|^{t-u-\varepsilon}|J_j|^{u+\varepsilon}. \tag{5}$$

If j satisfies ii), then for $i \in Q(j)$ we have

$$|I_i| = |I_i|^{t-u}|I_i|^u \geq 2^{-n(t-u)}|I_i|^u \geq c_{u,\varepsilon}^{-1} 2^{-m(t-u-\varepsilon)}|I_i|^u = c_{u,\varepsilon}^{-1}|D|^{t-u-\varepsilon}|I_i|^u.$$

Hence, summing over $i \in Q(j)$, we get

$$\begin{aligned} \sum_{i\in Q(j)} |I_i|^t &\geq c_{u,\varepsilon}^{-1}|D|^{t-u-\varepsilon} \sum_{i\in Q(j)} |I_i|^u \\ &\geq c_{u,\varepsilon}^{-1}|D|^{t-u-\varepsilon}\mathscr{M}^u_\infty(E \cap J_j) \geq |D|^{t-u-\varepsilon}|J_j|^{u+\varepsilon}. \end{aligned} \tag{6}$$

Combining (5) and (6) we get

$$\sum_{i=1}^{\infty} |I_i|^t \geq |D|^{t-u-\varepsilon} \sum_{j=1}^{k} |J_j|^{u+\varepsilon}$$
$$\geq |D|^{t-u-\varepsilon} \mathscr{M}_{\infty}^{u+\varepsilon}(D) = |D|^{t-u-\varepsilon}|D|^{u+\varepsilon} = |D|^u.$$

This shows that $E \in \mathscr{G}^u([0,1])$. Since u was arbitrary, the conclusion of the lemma follows. □

2.2. **Some Notations.** We will work with functions and probability measures on the interval $[0,1]$. For a function $f\colon [0,1] \to \mathbb{R}$ and $0 < s < 1$, we denote by $R_s f$ the function

$$R_s f(x) = \int |x-y|^{-s} f(y) \, \mathrm{d}y,$$

provided that the integral exists. Similarly, if μ is a measure on $[0,1]$, we let

$$R_s \mu(x) = \int |x-y|^{-s} \, \mathrm{d}\mu(y),$$

provided that the integral exists. This is the case, for instance, if f and the density of μ are in L^p for some $p > \frac{1}{1-s}$, since then Young's inequality implies, with $p > \frac{1}{1-s}$ and $q < \frac{1}{s}$ such that $1/p + 1/q \leq 1$, that

$$\|R_s f\|_\infty \leq \| |\cdot|^{-t} \|_q \|f\|_p < \infty.$$

2.3. **Some lemmata.** In this section, we prove some basic estimates that will be used in the proof of Theorem 1.

Lemma 2. *Let μ be a Borel measure on $[0,1]$. Assume that for some $s > 0$ holds*

$$C = \iint |x-y|^{-s} \, \mathrm{d}\mu(x)\mathrm{d}\mu(y) < \infty.$$

Then, if $M_m = \{\, (x,y) : |x-y|^{-s} > m \,\}$, we have for $0 < t < s$

$$\iint_{M_m} |x-y|^{-t} \, \mathrm{d}\mu(x)\mathrm{d}\mu(y) \leq C \frac{s}{s-t} m^{t/s-1}.$$

Proof. We note that $\iint |x-y|^{-t} \, \mathrm{d}\mu(x)\mathrm{d}\mu(y) < C$ for $t < s$ and that $M_m = \{\, (x,y) : |x-y|^{-t} > m^{t/s} \,\}$. Using that $\mu \times \mu(M_m) \leq C/m$ we get

$$\begin{aligned}
\iint_{M_m} |x-y|^{-t} \, \mathrm{d}\mu(x)\mathrm{d}\mu(y) &= \int_{m^{t/s}}^{\infty} \mu \times \mu(M_{u^{s/t}}) \, \mathrm{d}u + m^{t/s} \mu \times \mu(M_m) \\
&\leq \int_{m^{t/s}}^{\infty} \frac{C}{u^{s/t}} \, \mathrm{d}u + C m^{t/s-1} \\
&= C \frac{s}{s-t} m^{t/s-1}.
\end{aligned}$$ □

Corollary 1. *If μ_n are Borel measures on $[0,1]$ that converge weakly to a measure μ, and $\iint |x-y|^{-s} \, \mathrm{d}\mu_n(x)\mathrm{d}\mu_n(y)$ are uniformly bounded for some $s > 0$, then for $t < s$*

$$\iint |x-y|^{-t} \, \mathrm{d}\mu_n(x)\mathrm{d}\mu_n(y) \to \iint |x-y|^{-t} \, \mathrm{d}\mu(x)\mathrm{d}\mu(y)$$

as $n \to \infty$.

Proof. Let $\varepsilon > 0$ and $t < s$. Let M_m denote the same set as before. By Lemma 2 we can take m and N so large that

$$\iint_{M_m} |x-y|^{-t}\,\mathrm{d}\mu_n(x)\mathrm{d}\mu_n(y) < \varepsilon$$

for all $n > N$. Then

$$\iint |x-y|^{-t}\,\mathrm{d}\mu_n(x)\mathrm{d}\mu_n(y) \le \varepsilon + \iint \min\{|x-y|^{-t}, m^{t/s}\}\,\mathrm{d}\mu_n(x)\mathrm{d}\mu_n(y).$$

The function $\phi_m\colon (x,y) \mapsto \min\{|x-y|^{-t}, m^{t/s}\}$ is continuous, and so

$$\iint \phi_m\,\mathrm{d}\mu_n\mathrm{d}\mu_n \to \iint \phi_m\,\mathrm{d}\mu\mathrm{d}\mu \le \iint |x-y|^{-t}\,\mathrm{d}\mu(x)\mathrm{d}\mu(y),$$

as $n \to \infty$. Since ε is arbitrary this shows that

$$\limsup_{n\to\infty} \iint |x-y|^{-t}\,\mathrm{d}\mu_n(x)\mathrm{d}\mu_n(y) \le \iint |x-y|^{-t}\,\mathrm{d}\mu(x)\mathrm{d}\mu(y).$$

Now, the (obvious) inequality

$$\liminf_{n\to\infty} \iint |x-y|^{-t}\,\mathrm{d}\mu_n(x)\mathrm{d}\mu_n(y) \ge \iint |x-y|^{-t}\,\mathrm{d}\mu(x)\mathrm{d}\mu(y)$$

proves the corollary. □

Lemma 3. *Suppose $h\colon [0,1] \to \mathbb{R}$ is a non-negative function such that $R_s h$ is bounded. If $U \subset I \subset [0,1]$ are two intervals, then*

$$\frac{1}{|U|^s} \int_U \frac{h}{R_s(h|_I)}\,\mathrm{d}x \le 1$$

provided h does not vanish a.e. on I.

Proof. Let V be an interval. We first note that if $x \in V$, then

$$\frac{1}{1-s}|V|^{1-s} \le \int_V |x-y|^{-s}\,\mathrm{d}y \le \frac{2^s}{1-s}|V|^{1-s}. \tag{7}$$

Hence we have

$$|V|^{1-s}\chi_V(x) \le \frac{|V|^{1-s}}{1-s}\chi_V(x) \le \int |x-y|^{-s}\chi_V(y)\,\mathrm{d}y = R_s\chi_V(x) \tag{8}$$

for any x. For $x \notin V$ we have

$$R_s\chi_V(x) \ge d^{-s}|V|, \tag{9}$$

where $d = \sup\{\, |x-y| : y \in V \,\}$.

Assume that $h|_I$ is of the form

$$h|_I = \sum_{k\in J} c_k \chi_{I_k}, \tag{10}$$

where (I_k) are disjoint intervals that are subsets of I. Let $J_U \subset J$ be the set of indices such that I_k is a subset of U for $k \in J_U$. Then

$$R_s(h|_I) \ge \sum_{k\in J_U} c_k R_s \chi_{I_k}.$$

By (8) and (9), we have for $x \in I_l$ that

$$R_s(h|_I)(x) \ge c_l |I_l|^{1-s} + \sum_{J_U \ni k \ne l} c_k |U|^{-s} |I_k|,$$

and
$$\frac{h(x)}{R_s(h|_I)(x)} \leq \frac{c_l}{c_l|I_l|^{1-s} + \sum_{J_U \ni k \neq l} c_k|U|^{-s}|I_k|}.$$
Hence
$$\frac{1}{|U|^s}\int_U \frac{h(x)}{R_s(h|_I)(x)}\,\mathrm{d}x \leq |U|^{-s}\sum_{l\in J_U}\frac{c_l|I_l|}{c_l|I_l|^{1-s} + \sum_{J_U \ni k \neq l} c_k|U|^{-s}|I_k|}.$$
If we assume that $|I_k| = d$ for all k, then
$$\begin{aligned}\frac{1}{|U|^s}\int_U \frac{h(x)}{R_s(h|_I)(x)}\,\mathrm{d}x &\leq |U|^{-s}\sum_{l\in J_U}\frac{c_l d}{c_l d^{1-s} + \sum_{J_U \ni k \neq l} c_k|U|^{-s}d} \\ &= \sum_{l\in J_U}\frac{c_l|U|^{-s}d^s}{c_l + \sum_{J_U \ni k \neq l} c_k|U|^{-s}d^s} \\ &\leq \sum_{l\in J_U}\frac{c_l|U|^{-s}d^s}{c_l|U|^{-s}d^s + \sum_{J_U \ni k \neq l} c_k|U|^{-s}d^s} \leq 1.\end{aligned}$$

The general case is now proved by approximating with h of the form (10), with $|I_k| = |I_l|$. □

2.4. **Final part of the proof of Theorem 1.** We will prove that for any $t < s$ and $\varepsilon > 0$, there is a constant c such that
$$\liminf \mathscr{M}^t_\infty(E_n \cap I) \geq c|I|^{t+\varepsilon} \tag{11}$$
holds for any interval $I \subset [0,1]$. This implies according to Lemma 1, that E is in $\mathscr{G}^s([0,1])$.

Let $I \subset [0,1]$ be fixed and fix $t < s$. We denote by $\mu_n|_I$ the restriction of μ_n to I, i.e. $\mu_n|_I(A) = \mu_n(I \cap A)$. For large enough n we may assume that $\mu_n(I) > 0$.

We define new measures ν_n by
$$\nu_n(A) = \frac{\int_A (R_t\mu_n|_I)^{-1}\,\mathrm{d}\mu_n}{\int_I (R_t\mu_n|_I)^{-1}\,\mathrm{d}\mu_n}.$$
Clearly, the support of ν_n is equal to the support of $\mu_n|_I$.

We will prove that if n is large enough, then ν_n satisfies the estimates
$$\nu_n(U) \leq c\frac{|U|^t}{|I|^{t+\varepsilon}} \tag{12}$$
for each interval $U \subset I$, where c does not depend on I.

Suppose we have (12). Let $\{U_k\}$ be a cover of $I \cap E_n$ by disjoint intervals. Then
$$1 = \nu_n(\cup U_k) = \sum \nu_n(U_k) \leq \sum c\frac{|U_k|^t}{|I|^{t+\varepsilon}},$$
which implies that
$$\sum |U_k|^t \geq c^{-1}|I|^{t+\varepsilon}$$

holds for any cover $\{U_k\}$. This implies (11).

It remains to prove that (12) holds for large enough n. First, we see that (12) is equivalent to the inequality

$$\frac{1}{|U|^t}\int_U (R_t\mu_n|_I)^{-1}\,\mathrm{d}\mu_n \le \frac{c}{|I|^{t+\varepsilon}}\int_I (R_t\mu_n|_I)^{-1}\,\mathrm{d}\mu_n. \tag{13}$$

Consider the left side of (13). By Lemma 3 this is bounded by the constant 1, which is a constant that is independent of U, I and n.

The right side of (13) is estimated as

$$\int_I (R_t\mu_n|_I)^{-1}\frac{\mathrm{d}\mu_n}{\mu_n(I)} \ge \left(\int_I R_t\mu_n|_I \frac{\mathrm{d}\mu_n}{\mu_n(I)}\right)^{-1} = \frac{\mu_n(I)}{\int_I R_t\mu_n|_I\,\mathrm{d}\mu_n}.$$

Hence

$$\int_I (R_t\mu_n|_I)^{-1}\,\mathrm{d}\mu_n \ge \frac{\mu_n(I)^2}{\int_I R_t\mu_n|_I\,\mathrm{d}\mu_n}. \tag{14}$$

By Corollary 1 we have for $t < s$ that

$$\int_I R_t\mu_n|_I\,\mathrm{d}\mu_n \to \int_I R_t\mu|_I\,\mathrm{d}\mu,$$

as $n \to \infty$. Hence the right hand side of (14) converges to

$$\frac{\mu(I)^2}{\int_I R_t\mu|_I\,\mathrm{d}\mu}.$$

We want to prove that there exists a constant c, such that

$$\frac{c}{|I|^{t+\varepsilon}}\int_I (R_t\mu_n|_I)^{-1}\,\mathrm{d}\mu_n \ge 1,$$

holds for large enough n. To do so, it is sufficient to prove that

$$\frac{c}{|I|^{t+\varepsilon}}\frac{\mu(I)^2}{\int_I R_t\mu|_I\,\mathrm{d}\mu} \ge 1. \tag{15}$$

This is proved using (3) as follows. Let $g(u) = |u|^{-t}$ for $0 < |u| \le |I|$ and $g(u) = 0$ otherwise. Then

$$\int_I R_t\mu|_I\,\mathrm{d}\mu = \int_I\int_I g(x-y)h(y)\,\mathrm{d}y\,h(y)\,\mathrm{d}x = \int (g*(h\chi_I))(h\chi_I)\,\mathrm{d}x$$

Using Hölder's inequality and then Young's inequality we get

$$\int_I R_t\mu|_I\,\mathrm{d}\mu \le \|g*(h\chi_I)\|_2\|h\chi_I\|_2 \le \|g\|_1\|h\chi_I\|_2^2.$$

Since $\|g\|_1 = \frac{2}{1-t}|I|^{1-t}$, (3) implies that

$$\int_I R_t\mu|_I\,\mathrm{d}\mu \le \frac{2C_\varepsilon}{1-t}|I|^{-t-\varepsilon}\|h\chi_I\|_1^2 = \frac{2C_\varepsilon}{1-t}|I|^{-t-\varepsilon}\mu(I)^2.$$

It is now apparent that (15) holds if we choose $c > 2C_\varepsilon/(1-t)$.

This establishes (13), and hence finishes the proof.

3. Proof of Theorem 2

Here we will prove Theorem 2. The proof, that is based on Theorem 1, is divided into three parts, found in Sections 3.1, 3.2–3.3 and 3.4. We will construct measures that for almost all λ satisfies the assumptions of Theorem 1. In Section 3.1, we prove that the assumption (3) of Theorem 1 is satisfied, and in Sections 3.2 and 3.3 we prove that the assumption (4) of Theorem 1 is satisfied for almost all $\lambda \in (\frac{1}{2}, 0.64)$. Finally, in Section 3.4, we show how to conclude the desired result for almost all $\lambda \in (\frac{1}{2}, 1)$.

3.1. Some estimates on densities.

Let us now begin the proof of Theorem 2. According to Theorem 1, to prove that $E_\lambda(\alpha)$ is in $\mathscr{G}^s([0,1])$, it is sufficient to construct probability measures $\mu_{\lambda,k}$ with support in $E_{\lambda,k}(\alpha)$, converging weakly to a measure μ_λ with density h_λ in L^2, such that there exist constants C_ε and C with the property that

$$|I|^{1+\varepsilon} \|h_\lambda \chi_I\|_2^2 \leq C_\varepsilon \|h_\lambda \chi_I\|_1^2$$

holds for all intervals $I \subset [0,1]$, and

$$\iint |x-y|^{-s} \, \mathrm{d}\mu_{\lambda,k}(x) \mathrm{d}\mu_{\lambda,k}(y) \leq C$$

holds for infinitely many k. We will construct such measures for all $\lambda \in (\frac{1}{2}, 1)$, and prove that constants C_ε and C with the properties mentioned above, exists for almost all $\lambda \in (\frac{1}{2}, 1)$.

The measures $\mu_{\lambda,k}$ are constructed in the following way. We put

$$\Sigma_k = \{\, (a_0, a_1, \ldots, a_k) : a_n \in \{0,1\} \,\},$$

and define $\pi_k \colon \Sigma_k \to [0,1]$ by $\pi_k \colon (a_0, a_1, \ldots, a_k) \mapsto (1-\lambda) \sum_{n=0}^{k} a_n \lambda^n$. Hence $F_{\lambda,k} = \pi_k(\Sigma_k)$. Let ν denote the Lebesgue measure and let ν_I denote the normalised Lebesgue measure on an interval I. We denote by $B_r(x)$ the closed interval of length $2r$ and centre at x. Put

$$\mu_{\lambda,k} = 2^{-k} \sum_{a \in \Sigma_k} \nu_{B_{r_k}(\pi_k(a))}, \tag{16}$$

where $r_k = 2^{-\alpha k}$. Then $\mu_{\lambda,k}$ is a probability measure with support $E_{\lambda,k}(\alpha)$, and $\mu_{\lambda,k}$ converges weakly to a measure μ_λ as $k \to \infty$. The measure μ_λ is the distribution of the random Bernoulli convolution as described in [6], where it is proved that μ_λ has a density h_λ in L^2 for almost all $\lambda \in (\frac{1}{2}, 1)$. It gives positive measure to any interval $I \subset [0,1]$ with non-empty interior.

Let λ be such that h_λ has density in L^2. The density h_λ satisfies the functional equation

$$h_\lambda = \frac{1}{2\lambda} h_\lambda \circ S_1^{-1} + \frac{1}{2\lambda} h_\lambda \circ S_2^{-1}, \tag{17}$$

where S_1 and S_2 are the two contractions

$$\begin{aligned} S_1 &\colon x \mapsto \lambda x \\ S_2 &\colon x \mapsto \lambda x + 1 - \lambda. \end{aligned}$$

This can also be written in the following form. If I is an interval, and I_1, I_2 are two intervals such that $S_1(I_1) = I$ and $S_2(I_2) = I$, then

$$\begin{aligned}\int_I h_\lambda \, \mathrm{d}\nu &= \frac{1}{2\lambda}\int_I h_\lambda \circ S_1^{-1} \, \mathrm{d}\nu + \frac{1}{2\lambda}\int_I h_\lambda \circ S_2^{-1} \, \mathrm{d}\nu \\ &= \frac{1}{2}\int_{I_1} h_\lambda \, \mathrm{d}\nu + \frac{1}{2}\int_{I_2} h_\lambda \, \mathrm{d}\nu,\end{aligned}$$

or equivalently

$$\mu_\lambda(I) = \frac{1}{2}\mu_\lambda(S_1^{-1}(I)) + \frac{1}{2}\mu_\lambda(S_2^{-1}(I)). \tag{18}$$

We prove the following property of the measure μ_λ.

Proposition 1. *If λ is such that μ_λ has density h_λ in L^2, then for any $\varepsilon > 0$, there exists a constant C_ε such that*

$$|I|^{1+\varepsilon}\|h_\lambda \chi_I\|_2^2 \le C_\varepsilon \|h_\lambda \chi_I\|_1^2 \tag{19}$$

holds for any interval $I \subset [0,1]$.

To prove Proposition 1 we will need the following lemma.

Lemma 4. *Put $\theta = -\frac{\log 2}{\log \lambda}$ and fix $\lambda \in (\frac{1}{2}, 1)$. Then there is a constant K such that*

$$\frac{K}{2}(1-\lambda)^{-\theta} r^\theta \le \mu_\lambda([0,r)) \le 2K(1-\lambda)^{-\theta} r^\theta.$$

Moreover, there is a constant c such that for any interval $I \subset [0,1]$ of length r, holds

$$\mu_\lambda(I) \ge c r^\theta.$$

Proof. Let $V_0 = [0, 1-\lambda)$. Then $[0,1] \cap S_2^{-1}(V_0) = \emptyset$ and by (18), we have $K = \mu_\lambda(V_0)$, for some constant K.

Now, let V_k be defined recursively by $V_k = S_1(V_{k-1}) = [0, \lambda^k(1-\lambda))$. Then $\mu_\lambda(V_k) = 2^{-k}\mu_\lambda(V_0)$, so

$$\mu_\lambda(V_k) = K2^{-k}.$$

Consider the interval $[0,r)$, were $r \le 1-\lambda$. We let n be an integer such that

$$V_n \subset [0,r) \subset V_{n-1}.$$

Then

$$\frac{\log r - \log(1-\lambda)}{\log \lambda} \le n \le 1 + \frac{\log r - \log(1-\lambda)}{\log \lambda}.$$

We have

$$\mu_\lambda([0,r)) \ge \mu_\lambda(V_n) = K2^{-n} \ge \frac{K}{2}(1-\lambda)^{-\theta} r^\theta.$$

and

$$\mu_\lambda([0,r)) \le \mu_\lambda(V_{n-1}) = K2^{-n+1} \le 2K(1-\lambda)^{-\theta} r^\theta.$$

Let c_0 be the minimal μ_λ-measure of a sub-interval of $[0,1]$ of length $2\lambda - 1$. We have $c_0 > 0$. Consider an interval $I \subset [0,1]$. If $|I| < 2\lambda - 1$, then at least one of the intervals $S_1^{-1}(I)$ and $S_2^{-1}(I)$ are sub-intervals of $[0,1]$. So if $|I| = \lambda^k(2\lambda - 1)$ then there is an interval $I' \subset [0,1]$ of length $2\lambda - 1$, such that $I = S_{i_1} \circ \cdots \circ S_{i_k}(I')$. By (18) we have $\mu_\lambda(I) \ge 2^{-k}c_0$.

Finally, if $I \subset [0,1]$ is an arbitrary interval of length r we can choose an interval $J \subset I$ with $|J| = \lambda^n(2\lambda - 1)$, but $\lambda|I| \le |J|$. Since $\mu_\lambda(I) \ge \mu_\lambda(J) \ge$

$2^{-n}c_0 = c_0(2\lambda-1)^{-\theta}|J|^\theta \geq c_0(2\lambda-1)^{-\theta}\lambda^\theta r^\theta$, we conclude the theorem with $c = c_0(2\lambda-1)^{-\theta}\lambda^\theta$. $\square$

Proof of Proposition 1. Fix $\rho > 0$ and consider the intervals $I_x = B_\rho(x)$. The function

$$f(x) = \frac{\|h_\lambda \chi_{I_x}\|_2^2}{\|h_\lambda \chi_{I_x}\|_1^2}$$

defined on the interval $[0,1]$, is clearly continuous, and hence it is bounded on $[0,1]$. Moreover, using (17) one can prove that for $x = 0$ and $x = 1$, the function

$$g(r) = \frac{r\|h_\lambda \chi_{B_r(x)}\|_2^2}{\|h_\lambda \chi_{B_r(x)}\|_1^2}$$

is bounded when $r \to 0$. This is done in the following way. Assume that $x = 0$. The case $x = 1$ is similar by symmetry. Let r be fixed. We are going to estimate $g(\lambda r)$ in terms of $g(r)$. Let $J = [0, \lambda r]$. There are two intervals J_1 and J_2 such that $J = S_1(J_1) = S_2(J_2)$. We clearly have $J_1 = [0, r]$, and if r is sufficiently small, then $J_2 \cap [0,1] = \emptyset$. Therefore, we assume that r is so small that we have $J_2 \cap [0,1] = \emptyset$.

Now, $\int_{J_2} h_\lambda \, \mathrm{d}\nu = 0$ and (18) implies that

$$\int_J h_\lambda \, \mathrm{d}\nu = \frac{1}{2}\int_{J_1} h_\lambda \, \mathrm{d}\nu + \frac{1}{2}\int_{J_2} h_\lambda \, \mathrm{d}\nu = \frac{1}{2}\int_{J_1} h_\lambda \, \mathrm{d}\nu, \tag{20}$$

and by (17),

$$\int_J h_\lambda^2 \, \mathrm{d}\nu = \frac{1}{4\lambda^2}\int_J \big(h_\lambda \circ S_1^{-1} + h_\lambda \circ S_2^{-1}\big)^2 \, \mathrm{d}\nu = \frac{1}{4\lambda}\int_{J_1} h_\lambda^2 \, \mathrm{d}\nu. \tag{21}$$

By the definition of g we have

$$g(r) = r\frac{\int_{J_1} h_\lambda^2 \, \mathrm{d}\nu}{\big(\int_{J_1} h_\lambda \, \mathrm{d}\nu\big)^2}.$$

Hence, by (20) and (21),

$$g(\lambda r) = \lambda r \frac{\int_J h_\lambda^2 \, \mathrm{d}\nu}{\big(\int_J h_\lambda \, \mathrm{d}\nu\big)^2} = \lambda r \frac{\frac{1}{4\lambda}\int_{J_1} h_\lambda^2 \, \mathrm{d}\mu}{\frac{1}{4}\big(\int_{J_1} h_\lambda \, \mathrm{d}\nu\big)^2} = g(r).$$

By induction, we conclude that $g(\lambda^n r) = g(r)$ for all $n > 0$. It is moreover easy to see that g must be bounded on the interval $[\lambda\rho, \rho]$, and so g is bounded on $(0, \rho]$. (Indeed, g is continuous on any closed sub-interval of $(0,1)$.)

We have proved that both f and g are bounded. Let C_0 be a constant such that $f \leq C_0/(2\rho) = C_0/|I_x|$ on $[-\rho, 1+\rho]$. This means that we have

$$|I|\|h_\lambda \chi_I\|_2^2 \leq C_0 \|h_\lambda \chi_I\|_1^2 \tag{22}$$

for all intervals $I \subset [0,1]$ of length 2ρ. We also let C_1 be a constant such that $g \leq C_1$. Hence we have

$$r\|h_\lambda \chi_{[0,r)}\|_2^2 \leq C_1 \|h_\lambda \chi_{[0,r)}\|_1^2 \tag{23}$$

for all $0 < r < 1$. By symmetry of μ_λ, we have the same inequality for the intervals $(1-r, 1]$.

We will now proceed by induction in the following way. Suppose (22) holds for all intervals $I \subset [0,1]$ of a certain length $L < 1$. Let $J \subset [0,1]$ be an interval with $|J| = \lambda L$. We want to prove that

$$|J| \|h_\lambda \chi_J\|_2^2 \leq C_2 \|h_\lambda \chi_J\|_1^2.$$

There are two intervals J_1 and J_2 such that $J = S_1(J_1) = S_2(J_2)$. By (17) we have

$$\int_J h_\lambda \,\mathrm{d}\nu = \frac{1}{2}\int_{J_1} h_\lambda \,\mathrm{d}\nu + \frac{1}{2}\int_{J_2} h_\lambda \,\mathrm{d}\nu, \tag{24}$$

and

$$\begin{aligned}\int_J h_\lambda^2 \,\mathrm{d}\nu &= \frac{1}{4\lambda^2}\int_J \big(h_\lambda \circ S_1^{-1} + h_\lambda \circ S_2^{-1}\big)^2 \,\mathrm{d}\nu \\ &= \frac{1}{4\lambda}\int_{J_1} h_\lambda^2 \,\mathrm{d}\nu + \frac{1}{4\lambda}\int_{J_2} h_\lambda^2 \,\mathrm{d}\nu + \frac{1}{2\lambda}\int_{J_1} h_\lambda \cdot h_\lambda \circ T \,\mathrm{d}\nu,\end{aligned}$$

where $T = S_2^{-1} \circ S_1$ is a translation. By the Cauchy–Bunyakovsky–Schwarz inequality we have

$$\int_{J_1} h_\lambda \cdot h_\lambda \circ T \,\mathrm{d}\nu \leq \Big(\int_{J_1} h_\lambda^2 \,\mathrm{d}\nu \int_{J_2} h_\lambda^2 \,\mathrm{d}\nu\Big)^{\frac{1}{2}},$$

and so

$$\begin{aligned}\int_J h_\lambda^2 \,\mathrm{d}\nu &\leq \frac{1}{4\lambda}\int_{J_1} h_\lambda^2 \,\mathrm{d}\nu + \frac{1}{4\lambda}\int_{J_2} h_\lambda^2 \,\mathrm{d}\nu + \frac{1}{2\lambda}\Big(\int_{J_1} h_\lambda^2 \,\mathrm{d}\nu \int_{J_2} h_\lambda^2 \,\mathrm{d}\nu\Big)^{\frac{1}{2}} \\ &= \frac{1}{4\lambda}\bigg(\Big(\int_{J_1} h_\lambda^2 \,\mathrm{d}\nu\Big)^{\frac{1}{2}} + \Big(\int_{J_2} h_\lambda^2 \,\mathrm{d}\nu\Big)^{\frac{1}{2}}\bigg)^2.\end{aligned}$$

We want to use (22) on each of the integrals $\int_{J_1} h_\lambda^2 \,\mathrm{d}\nu$ and $\int_{J_2} h_\lambda^2 \,\mathrm{d}\nu$ above, but it may happen that one of J_1 and J_2 are not a subset of $[0,1]$. Assume therefore that $J_1 \subset [0,1]$ and that J_2 is not necessarily a subset of $[0,1]$. (The case $J_2 \subset [0,1]$ and J_1 is not necessarily a subset of $[0,1]$ is analogous.) Let $\tilde{J}_2$ be the intersection $\tilde{J}_2 = J_2 \cap [0,1]$. We have $0 \leq |\tilde{J}_2| \leq |J_1|$. Our estimate (23) implies that

$$\int_{J_2} h_\lambda^2 \,\mathrm{d}\nu = \int_{\tilde{J}_2} h_\lambda^2 \,\mathrm{d}\nu \leq \frac{C_1}{|\tilde{J}_2|}\Big(\int_{\tilde{J}_2} h_\lambda \,\mathrm{d}\nu\Big)^2 = \frac{C_1}{|\tilde{J}_2|}\Big(\int_{J_2} h_\lambda \,\mathrm{d}\nu\Big)^2,$$

and (22) implies that

$$\int_{J_1} h_\lambda^2 \,\mathrm{d}\nu \leq \frac{C_0}{|J_1|}\Big(\int_{J_1} h_\lambda \,\mathrm{d}\nu\Big)^2.$$

Hence

$$\begin{aligned}\int_J h_\lambda^2 \, \mathrm{d}\nu &\leq \frac{1}{4\lambda}\biggl(\Bigl(\int_{J_1} h_\lambda^2 \, \mathrm{d}\nu\Bigr)^{\frac{1}{2}} + \Bigl(\int_{J_2} h_\lambda^2 \, \mathrm{d}\nu\Bigr)^{\frac{1}{2}}\biggr)^2 \\ &\leq \frac{1}{4\lambda}\biggl(\frac{C_0^{\frac{1}{2}}}{|J_1|^{\frac{1}{2}}} \int_{J_1} h_\lambda \, \mathrm{d}\nu + \frac{C_1^{\frac{1}{2}}}{|\tilde{J}_2|^{\frac{1}{2}}} \int_{\tilde{J}_2} h_\lambda \, \mathrm{d}\nu\biggr)^2 \\ &= \frac{1}{\lambda}\frac{1}{|J_1|}\left(\frac{C_0^{\frac{1}{2}} \int_{J_1} h_\lambda \, \mathrm{d}\nu + C_1^{\frac{1}{4}} \frac{|J_1|^{\frac{1}{2}}}{|\tilde{J}_2|^{\frac{1}{2}}} \int_{\tilde{J}_2} h_\lambda \, \mathrm{d}\nu}{\int_{J_1} h_\lambda \, \mathrm{d}\nu + \int_{\tilde{J}_2} h_\lambda \, \mathrm{d}\nu}\right)^2 \Bigl(\int_J h_\lambda \, \mathrm{d}\nu\Bigr)^2. \qquad (25)\end{aligned}$$

We want to bound

$$Q = \frac{C_0^{\frac{1}{2}} \int_{J_1} h_\lambda \, \mathrm{d}\nu + C_1^{\frac{1}{4}} \frac{|J_1|^{\frac{1}{2}}}{|\tilde{J}_2|^{\frac{1}{2}}} \int_{\tilde{J}_2} h_\lambda \, \mathrm{d}\nu}{\int_{J_1} h_\lambda \, \mathrm{d}\nu + \int_{\tilde{J}_2} h_\lambda \, \mathrm{d}\nu},$$

and note that it is a weighted average of $C_0^{\frac{1}{2}}$ and $C_1^{\frac{1}{2}} \frac{|J_1|^{\frac{1}{2}}}{|\tilde{J}_2|^{\frac{1}{2}}}$. Let $d = |J_1|$ and $e = |\tilde{J}_2|^{\frac{1}{2}}$. Then $0 \leq e \leq d$. If we take C_0 much larger than C_1, we may conclude, by Lemma 4 and the fact that Q is a weighted average, that

$$\begin{aligned}Q &\leq \frac{C_0^{\frac{1}{2}} c d^\theta + C_1^{\frac{1}{2}} \frac{|J_1|^{\frac{1}{2}}}{|\tilde{J}_2|^{\frac{1}{2}}} \frac{K}{2} (1-\lambda)^{-\theta} e^\theta}{c d^\theta + \frac{K}{2}(1-\lambda)^{-\theta} e^\theta} \\ &= \frac{C_0^{\frac{1}{2}} c (1-\lambda)^\theta + C_1^{\frac{1}{2}} \frac{K}{2} \bigl(\frac{e}{d}\bigr)^{\theta - \frac{1}{2}}}{c(1-\lambda)^\theta + \frac{K}{2}\bigl(\frac{e}{d}\bigr)^\theta} \leq C_0^{\frac{1}{2}} \eta, \qquad (26)\end{aligned}$$

where

$$\eta = \sup_{0 \leq t \leq 1} \frac{2c(1-\lambda)^\theta + (C_1/C_0)^{\frac{1}{2}} K t^{\theta - \frac{1}{2}}}{2c(1-\lambda)^\theta + K t^\theta}.$$

Combining (25) and (26), we get that

$$\int_J h_\lambda^2 \, \mathrm{d}\nu \leq \frac{C_0 \eta^2}{|J|} \Bigl(\int_J h_\lambda \, \mathrm{d}\nu\Bigr)^2. \qquad (27)$$

Hence we have determined that if (22) holds for all intervals of a fixed size L, then (27) holds for intervals of length λL. By induction, starting with (22) for intervals of length 2ρ, we conclude that

$$\int_J h_\lambda^2 \, \mathrm{d}\nu \leq \frac{C_0 \eta^{2n}}{|J|} \Bigl(\int_J h_\lambda \, \mathrm{d}\nu\Bigr)^2 \qquad (28)$$

holds for any interval of length $2\lambda^n \rho$. This is not yet quite what we want. However, by choosing C_0 large, we can make η arbitrarily close to 1. In this way, for any ε, we will achieve the estimate

$$\int_J h_\lambda^2 \, \mathrm{d}\nu \leq \frac{C_\varepsilon}{|J|^{1+\varepsilon}} \Bigl(\int_J h_\lambda \, \mathrm{d}\nu\Bigr)^2$$

for any interval of length $2\lambda^n \rho$. Since C_0 does not blow up when we change ρ a bit, we conclude (19) for intervals of any length. □

3.2. **Some estimates using Fourier analysis.** We let $\mu_{\lambda,k}$ be the measures defined in the previous section. To emphasise the dependence on α, which will prove important in this section, we denote $\mu_{\lambda,k}$ by $\mu_{\alpha,\lambda,k}$ and we let $h_{\alpha,\lambda,k}$ denote the densities of the measures $\mu_{\alpha,\lambda,k}$. We are interested in determining for which α, λ and s, there is a constant C such that

$$\iint |x-y|^{-s}\, \mathrm{d}\mu_{\alpha,\lambda,k}(x)\mathrm{d}\mu_{\alpha,\lambda,k}(y) < C,$$

holds for infinitely many k. In this section, will prove the following proposition.

Proposition 2. *Let $\alpha s < 1$. If $\lambda \in (\frac{1}{2}, 0.64)$, then, almost surely, there exists a constant C such that*

$$\iint |x-y|^{-s}\, \mathrm{d}\mu_{\alpha,\lambda,k}(x)\mathrm{d}\mu_{\alpha,\lambda,k}(y) \leq C, \tag{29}$$

holds for infinitely many k.

Proposition 2 implies together with Proposition 1 the statement of Theorem 2 for almost all $\lambda \in (\frac{1}{2}, 0.64)$.

We will estimate the integrals in Proposition 2 using Fourier transforms. We use the convention that the Fourier transform of a function f is the function

$$\hat{f}(\xi) = \int e^{-i2\pi\xi x} f(x)\, \mathrm{d}x.$$

Writing as before, $R_s h(x) = |\cdot|^{-s} * h(x) = \int |x-y|^{-s} h(y)\, \mathrm{d}y$, we have

$$\begin{aligned}\iint |x-y|^{-s}\, \mathrm{d}\mu_{\alpha,\lambda,k}(x)\mathrm{d}\mu_{\alpha,\lambda,k}(y) &= \int h_{\alpha,\lambda,k}(x) R_s h_{\alpha,\lambda,k}(x)\, \mathrm{d}x \\ &= \int \overline{\hat{h}_{\alpha,\lambda,k}(\xi)}\, \widehat{R_s h_{\alpha,\lambda,k}}(\xi)\, \mathrm{d}\xi = c_s \int |\hat{h}_{\alpha,\lambda,k}(\xi)|^2 |\xi|^{s-1}\, \mathrm{d}\xi,\end{aligned}$$

where c_s is a constant depending only on s. We are going to estimate $\int |\hat{h}_{\alpha,\lambda,k}(\xi)|^2 |\xi|^{s-1}\, \mathrm{d}\xi$.

To determine the Fourier transform of $h_{\alpha,\lambda,k}$ we note that the Fourier transform of the measure $\frac{1}{2}(\delta_a + \delta_0)$ is $e^{-i\pi a\xi}\cos(\pi a\xi)$. The measure $\mu_{\alpha,\lambda,k}$ is the convolution of the measures $\frac{1}{2}(\delta_{\lambda^n} + \delta_0)$, $n = 0, 1, \ldots, k$, and the uniform mass-distribution on the interval $[-2^{-\alpha k}, 2^{-\alpha k}]$. Hence we have

$$\hat{h}_{\alpha,\lambda,k}(\xi) = \frac{\phi_{\lambda,k}(\xi)}{2\pi} \frac{\sin(2^{-\alpha k}\xi)}{2^{-\alpha k}\xi} \prod_{n=0}^{k} \cos(\pi\lambda^n\xi), \tag{30}$$

and

$$\hat{h}_{\alpha,\lambda}(\xi) = \phi_\lambda(\xi) \prod_{n=0}^{\infty} \cos(\pi\lambda^n\xi),$$

where $|\phi_{\lambda,k}(\xi)| = |\phi_\lambda(\xi)| = 1$. We also introduce the related function

$$g_{\alpha,\lambda,k}(\xi) = \eta_{\alpha,k}(\xi) \prod_{n=0}^{k} \cos(\pi\lambda^n\xi), \tag{31}$$

where

$$\eta_{\alpha,k}(\xi) = \begin{cases} 1 & \text{if } |\xi| \leq 2^{\alpha k}, \\ \frac{2^{\alpha k}}{\xi} & \text{if } |\xi| > 2^{\alpha k}. \end{cases}$$

It appears from (30) and (31) that $2\pi|\hat{h}_{\alpha,\lambda,k}| \leq |g_{\alpha,\lambda,k}|$. Hence,

$$\int |\hat{h}_{\alpha,\lambda,k}(\xi)|^2 |\xi|^{s-1} \, \mathrm{d}\xi \leq \int |g_{\alpha,\lambda,k}(\xi)|^2 |\xi|^{s-1} \, \mathrm{d}\xi = 2 \int_0^\infty |g_{\alpha,\lambda,k}(\xi)|^2 |\xi|^{s-1} \, \mathrm{d}\xi,$$

Moreover, instead of estimating $\int |g_{\alpha,\lambda,k}(\xi)|^2 |\xi|^{s-1} \, \mathrm{d}\xi$, we can do a bit more, and instead estimate $\int |g_{\alpha,\lambda,k}(\xi)|^4 |\xi|^{2s-1} \, \mathrm{d}\xi$.

Proposition 3. *For almost all $\lambda \in [\frac{1}{2}, 0.64]$ there are constants C and D such that*

$$\int |g_{\alpha,\lambda,k}(\xi)|^4 |\xi|^{2s-1} \, \mathrm{d}\xi < C 4^{(\alpha s - 1)k} + D,$$

holds for infinitely many k.

Since $|g_{\alpha,\lambda,k}(\xi)| \leq 1$, we can conclude from Proposition 3, that if $\alpha s < 1$, then for almost all $\lambda \in [\frac{1}{2}, 0.64]$ there is a constant C such that

$$\int |\hat{h}_{\alpha,\lambda,k}(\xi)|^2 |\xi|^{s-1} \, \mathrm{d}\xi \leq \int |g_{\alpha,\lambda,k}(\xi)|^2 |\xi|^{s-1} \, \mathrm{d}\xi < C,$$

holds for infinitely many k. Hence, Proposition 3 implies Proposition 2.

In fact, as we shall see in Section 3.4, Proposition 3 implies more, and will be important to get the desired result not only for λ in $[\frac{1}{2}, 0.64]$, but the entire interval $[\frac{1}{2}, 1]$.

Remark 2. We have defined the measures $\mu_{\alpha,\lambda,k}$ so that their densities are normalised sums of indicator functions of intervals with radius $2^{-\alpha k}$ and centres in $F_{\lambda,k}$. Let $c > 0$. The proof of Proposition 3 will work without changes, if we instead would have defined the measures $\mu_{\alpha,\lambda,k}$ such that their densities where based on intervals of radius $c2^{-\alpha k}$ instead of $2^{-\alpha k}$.

In Section 3.4, we will make use of this somewhat more general version of Proposition 3, and we will then denote the corresponding measures by $\mu_{\alpha,\lambda,k,c}$, but to make the notations less heavy, we will only prove Proposition 3 in the case $c = 1$.

3.3. **Proof of Proposition 3.** We write the interval $[0, \infty)$ as the disjoint union $[0, \infty) = I_1(k) \cup I_2(k) \cup I_3(k)$, where

$$I_1(k) = [0, 1), \quad I_2(k) = [1, 2^{\alpha k}), \quad \text{and} \quad I_3(k) = [2^{\alpha k}, \infty),$$

and treat separately the integrals

$$J_i(\lambda) = \int_{I_i(k)} |g_{\alpha,\lambda,k}(\xi)|^4 |\xi|^{2s-1} \, \mathrm{d}\xi, \quad i = 1, 2, 3.$$

On the intervals $I_1(k)$ and $I_2(k)$ we have trivially that

$$\eta_{\alpha,k}(\xi)^4 = 1, \tag{32}$$

and on the interval $I_3(k)$, we have

$$\eta_{\alpha,k}(\xi)^4 = \frac{2^{4\alpha k}}{\xi^4}. \tag{33}$$

Let us start by estimating $J_1(\lambda)$. By (32) we get that

$$J_1(\lambda) = \int_{I_1(k)} |g_{\alpha,\lambda,k}(\xi)|^4 |\xi|^{2s-1}\,\mathrm{d}\xi \le \int_{I_1(k)} |\xi|^{2s-1}\,\mathrm{d}\xi = \frac{1}{2s}. \tag{34}$$

Next, we estimate, using (32) and (33), that

$$\begin{aligned}
J_2(\lambda) &= \int_{I_2(k)} |g_{\alpha,\lambda,k}(\xi)|^4 |\xi|^{2s-1}\,\mathrm{d}\xi \le \int_1^{2^{\alpha k}} \Big(\prod_{n=0}^{k} \cos^2(\lambda^n \xi)\Big)^2 |\xi|^{2s-1}\,\mathrm{d}\xi,\\
J_3(\lambda) &= \int_{I_3(k)} |g_{\alpha,\lambda,k}(\xi)|^4 |\xi|^{2s-1}\,\mathrm{d}\xi \le \int_{2^{\alpha k}}^{\infty} \Big(\prod_{n=0}^{k} \cos^2(\lambda^n \xi)\Big)^2 2^{4\alpha k}|\xi|^{2s-5}\,\mathrm{d}\xi.
\end{aligned}$$

We write

$$P_k(\lambda,\xi) = \prod_{n=0}^{k} \cos^2(\lambda^n\xi) = \frac{1}{4^{k+1}} \sum_{a,b\in\Sigma_k} \cos(\sum_{n=0}^{k}(a_n - b_n)\lambda^n\xi),$$

and put $\theta_{a,b}(\lambda) = \sum_{n=0}^{k}(a_n - b_n)\lambda^n$. Define $p_t\colon [1,\infty) \to \mathbb{R}$, such that $p_t(\xi) = n^t$ for $n \le \xi < n+1$. Then, if $t < 0$, we have $p_t(\xi) \ge \xi^t$, and therefore

$$\begin{aligned}
J_2(\lambda) &\le \frac{1}{4^{k+1}} \sum_{a,b\in\Sigma_k} \int_1^{2^{\alpha k}} \cos(\theta_{a,b}(\lambda)\xi) P_k(\lambda,\xi) p_{2s-1}(\xi)\,\mathrm{d}\xi,\\
J_3(\lambda) &\le \frac{16^{\alpha k}}{4^{k+1}} \sum_{a,b\in\Sigma_k} \int_{2^{\alpha k}}^{\infty} \cos(\theta_{a,b}(\lambda)\xi) P_k(\lambda,\xi) p_{2s-5}(\xi)\,\mathrm{d}\xi.
\end{aligned}$$

Hence

$$\begin{aligned}
J_2(\lambda) &\le \frac{1}{4^{k+1}} \sum_{a,b\in\Sigma_k} \sum_{m=1}^{2^{\alpha k}} \int_m^{m+1} \cos(\theta_{a,b}(\lambda)\xi) P_k(\lambda,\xi) m^{2s-1}\,\mathrm{d}\xi,\\
J_3(\lambda) &\le \frac{16^{\alpha k}}{4^{k+1}} \sum_{a,b\in\Sigma_k} \sum_{m=2^{\alpha k}}^{\infty} \int_m^{m+1} \cos(\theta_{a,b}(\lambda)\xi) P_k(\lambda,\xi) m^{2s-5}\,\mathrm{d}\xi.
\end{aligned}$$

If a and b are two different elements in Σ_k, then $\theta_{a,b}(\lambda) \ne 0$, except for finitely many λ. Therefore, for $a \ne b$, and almost all λ, we have

$$\begin{aligned}
\int_m^{m+1} \cos(\theta_{a,b}(\lambda)\xi) P_k(\lambda,\xi) m^{2s-1}\,\mathrm{d}\xi &\le \int_m^{m+1} \cos(\theta_{a,b}(\lambda)\xi) m^{2s-1}\,\mathrm{d}\xi\\
&= \Big(\frac{\sin(\theta_{a,b}(\lambda)(m+1))}{\theta_{a,b}(\lambda)} - \frac{\sin(\theta_{a,b}(\lambda)m)}{\theta_{a,b}(\lambda)}\Big) m^{2s-1},
\end{aligned}$$

and we can thus write

$$\begin{aligned}
&\sum_{m=1}^{2^{\alpha k}} \int_m^{m+1} \cos(\theta_{a,b}(\lambda)\xi) P_k(\lambda,\xi) m^{2s-1} \,\mathrm{d}\xi \mathrm{d}\lambda \\
&\quad \leq \sum_{m=1}^{2^{\alpha k}} \int_m^{m+1} \cos(\theta_{a,b}(\lambda)\xi) m^{2s-1} \,\mathrm{d}\xi \mathrm{d}\lambda \\
&\quad = \sum_{m=2}^{1+2^{\alpha k}} \frac{\sin(\theta_{a,b}(\lambda)m)}{\theta_{a,b}(\lambda)} (m-1)^{2s-1} - \sum_{m=1}^{2^{\alpha k}} \frac{\sin(\theta_{a,b}(\lambda)m)}{g_{a,b}(\lambda)} m^{2s-1} \\
&\quad = \frac{\sin(\theta_{a,b}(\lambda)(1+2^{\alpha k})}{\theta_{a,b}(\lambda)} (2^{\alpha k})^{2s-1} - \frac{\sin(\theta_{a,b}(\lambda)}{\theta_{a,b}(\lambda)} \\
&\qquad + \sum_{m=2}^{2^{\alpha k}} \frac{\sin(\theta_{a,b}(\lambda)m}{\theta_{a,b}(\lambda)} \big((m-1)^{2s-1} - m^{2s-1}\big).
\end{aligned}$$

We now consider $[p,q] \subset (\frac{1}{2}, 0.64)$, and estimate $\int_p^q J_2(\lambda)\,\mathrm{d}\lambda$. For this purpose we will use the following lemma. To state it, we use the notation $l(a,b)$ to denote the smallest integer l such that $a_l \neq b_l$ if a and b are two different elements of Σ_k. We also put

$$J_{a,b,m} = \int_p^q \frac{\sin(\theta_{a,b}(\lambda)m)}{\theta_{a,b}(\lambda)} \,\mathrm{d}\lambda.$$

Lemma 5. *There is a constant K_1 such that for all $a, b \in \Sigma_k$, with $a \neq b$, and all m,*

$$|J_{a,b,m}| \leq K_1 p^{-l(a,b)}.$$

It is intuitively clear that Lemma 5 follows from Solomyak's transversality lemma [6]. Details on how to prove this, are available in [6], where it is shown how it follows from a lemma in [4], that is called Lemma 2.2 in [6].

By a change of order of integration we have that

$$\int_p^q J_2(\lambda)\,\mathrm{d}\lambda \leq L_1 + L_2,$$

where

$$\begin{aligned}
L_1 = \frac{1}{4^{k+1}} \sum_{l=0}^{k} \sum_{\substack{a,b\in\Sigma_{k,l} \\ l(a,b)=l}} \Bigg(&\sum_{m=2}^{2^{\alpha k}} J_{a,b,m}\big((m-1)^{2s-1} - m^{2s-1}\big) \\
&+ J_{a,b,1+2^{\alpha k}} (2^{\alpha k})^{2s-1} - J_{a,b,1} \Bigg),
\end{aligned}$$

$$L_2 = \frac{1}{4^{k+1}} \sum_{a=b\in\Sigma_k} \sum_{m=1}^{2^{\alpha k}} \int_p^q \int_m^{m+1} \cos(\theta_{a,b}(\lambda)\xi) P_k(\lambda,\xi) m^{2s-1} \,\mathrm{d}\xi \mathrm{d}\lambda.$$

The first part is estimated with use of Lemma 5. We have

$$\begin{aligned}
L_1 &\leq \frac{1}{4^{k+1}} \sum_{l=0}^{k} \sum_{a,b\in\Sigma_{k,l}} \frac{K_1}{p^l} \biggl(\sum_{m=2}^{2^{\alpha k}} \bigl((m-1)^{2s-1} - m^{2s-1}\bigr) + (2^{\alpha k})^{2s-1} + 1 \biggr), \\
&= \frac{1}{4^{k+1}} \sum_{l=0}^{k} 2^{k+1} 2^{k+1-l} 2K_1 p^{-l} \leq \frac{2K_1}{1-(2p)^{-1}}.
\end{aligned}$$

We now turn to the estimate of L_2. If $a = b$, then $\theta_{a,b} = 0$, and so

$$\begin{aligned}
L_2 &= \frac{1}{2^{k+1}} \sum_{m=1}^{2^{\alpha k}} \int_p^q \int_m^{m+1} P_k(\lambda,\xi) m^{2s-1} \,\mathrm{d}\xi \mathrm{d}\lambda \\
&= \frac{1}{6^{k+1}} \sum_{c,d\in\Sigma_k} \sum_{m=1}^{2^{\alpha k}} \int_p^q \int_m^{m+1} \cos(\theta_{c,d}(\lambda)\xi) m^{2s-1} \,\mathrm{d}\xi \mathrm{d}\lambda \\
&= \frac{1}{2^{k+1}} L_1 + \frac{1}{6^{k+1}} \sum_{c=d\in\Sigma_k} \sum_{m=1}^{2^{\alpha k}} \int_p^q \int_m^{m+1} \cos(\theta_{c,d}(\lambda)\xi) m^{2s-1} \,\mathrm{d}\xi \mathrm{d}\lambda \\
&\leq \frac{1}{2^{k+1}} L_1 + \frac{1}{6^{k+1}} 2^{k+1} \sum_{m=1}^{2^{\alpha k}} (q-p) m^{2s-1} \\
&\leq \frac{1}{2^{k+1}} L_1 + \frac{1}{4^{k+1}} K_2 2^{2\alpha s k},
\end{aligned}$$

where K_2 is a constant that does not depend on k.

Putting the estimates of L_1 and L_2 together, we find that

$$\int_p^q J_2(\lambda)\,\mathrm{d}\lambda \leq \frac{3K_1}{1-(2p)^{-1}} + \frac{K_2}{4} 2^{(2\alpha s - 2)k} \leq K_3(1 + 2^{(2\alpha s-2)k}), \tag{35}$$

where K_3 is a constant that does not depend on k.

We will now estimate $J_3(\lambda)$. In the same way as for J_2, we have that

$$\int_p^q J_3(\lambda)\,\mathrm{d}\lambda \leq M_1 + M_2,$$

where

$$\begin{aligned}
M_1 = \frac{16^{\alpha k}}{4^{k+1}} \sum_{l=0}^{k} \sum_{\substack{a,b\in\Sigma_{k,l} \\ l(a,b)=l}} \biggl(&\sum_{m=2^{\alpha k}}^{\infty} J_{a,b,m}\bigl((m-1)^{2s-5} - m^{2s-5}\bigr) \\
&+ J_{a,b,1+2^{\alpha k}} (2^{\alpha k})^{2s-5} - J_{a,b,1} \biggr),
\end{aligned}$$

$$M_2 = \frac{16^{\alpha k}}{4^{k+1}} \sum_{a=b\in\Sigma_k} \sum_{m=2^{\alpha k}}^{\infty} \int_p^q \int_m^{m+1} \cos(\theta_{a,b}(\lambda)\xi) P_k(\lambda,\xi) m^{2s-5} \,\mathrm{d}\xi \mathrm{d}\lambda.$$

The first part is again estimated with use of Lemma 5. We have

$$\begin{aligned} M_1 &\leq \frac{16^{\alpha k}}{4^{k+1}} \sum_{l=0}^{k} \sum_{a,b \in \Sigma_{k,l}} \frac{K_1}{p^l} \Biggl(\sum_{m=2^{\alpha k}}^{\infty} \bigl((m-1)^{2s-5} - m^{2s-5}\bigr) + (2^{\alpha k})^{2s-5} + 1 \Biggr), \\ &= \frac{16^{\alpha k}}{4^{k+1}} \sum_{l=0}^{k} 2^{k+1} 2^{k+1-l} 2 K_1 p^{-l} \leq \frac{2K_1}{1-(2p)^{-1}}. \end{aligned}$$

We proceed with the estimate of M_2. If $a = b$, then $\theta_{a,b} = 0$, and so

$$\begin{aligned} M_2 &= \frac{16^{\alpha k}}{2^{k+1}} \sum_{m=2^{\alpha k}}^{\infty} \int_p^q \int_m^{m+1} P_k(\lambda, \xi) m^{2s-5} \, \mathrm{d}\xi \mathrm{d}\lambda \\ &= \frac{16^{\alpha k}}{6^{k+1}} \sum_{c,d \in \Sigma_k} \sum_{m=2^{\alpha k}}^{\infty} \int_p^q \int_m^{m+1} \cos(\theta_{c,d}(\lambda)\xi) m^{2s-5} \, \mathrm{d}\xi \mathrm{d}\lambda \\ &= \frac{1}{2^{k+1}} M_1 + \frac{16^{\alpha k}}{6^{k+1}} \sum_{c=d \in \Sigma_k} \sum_{m=2^{\alpha k}}^{\infty} \int_p^q \int_m^{m+1} \cos(\theta_{c,d}(\lambda)\xi) m^{2s-5} \, \mathrm{d}\xi \mathrm{d}\lambda \\ &\leq \frac{1}{2^{k+1}} M_1 + \frac{16^{\alpha k}}{6^{k+1}} 2^{k+1} \sum_{m=2^{\alpha k}}^{\infty} (q-p) m^{2s-5} \\ &\leq \frac{1}{2^{k+1}} M_1 + \frac{1}{4^{k+1}} K_4 2^{2\alpha s k}. \end{aligned}$$

where K_4 does not depend on k.

The estimates of M_1 and M_2 imply that

$$\int_p^q J_3(\lambda) \, \mathrm{d}\lambda \leq \frac{3K_1}{1-(2p)^{-1}} + K_4 2^{(2s\alpha-2)k} \leq K_5 (1 + 2^{(2\alpha s-2)k}), \tag{36}$$

where K_5 is a constant.

From (34), (35) and (36), we conclude that

$$\int_p^q \int |\hat{h}_{\lambda,k}(\xi)|^4 |\xi|^{2s-1} \, \mathrm{d}\xi \mathrm{d}\lambda \leq \frac{1}{s} + 2(K_3 + K_5)(1 + 2^{(2\alpha s-2)k}).$$

Hence, for almost all $\lambda \in [p,q]$, there are constants $C = C(\lambda)$ and $D = D(\lambda)$ such that

$$\int |\hat{h}_{\lambda,k}(\xi)|^4 |\xi|^{2s-1} \, \mathrm{d}\xi < C 2^{2(\alpha s-1)k} + D,$$

holds for infinitely many k. Since p and q are arbitrary this proves Proposition 3.

3.4. **Convolutions.** We have proved the statement of Theorem 2 for almost all $\lambda \in (\frac{1}{2}, 0.64)$. In this section we are going to show the result for almost all $\lambda \in (\frac{1}{2}, 1)$.

Let λ be such that $\lambda^2 \in (\frac{1}{2}, 0.64)$. We define the measure $\mu^{(2)}_{\alpha,\lambda,2k+1}$ as the convolution of $\mu_{2\alpha,\lambda^2,k,c}$ and $\mu_{2\alpha,\lambda^2,k,c} \circ S_1^{-1}$, where S_1 is the contraction $S_1(x) = \lambda x$. Hence, if we let $h^{(2)}_{\alpha,\lambda,2k+1}$ denote the density of $\mu^{(2)}_{\alpha,\lambda,2k+1}$, we have $h^{(2)}_{\alpha,\lambda,2k+1} = h_{2\alpha,\lambda^2,k,c} * (h_{2\alpha,\lambda^2,k,c} \circ S_1^{-1})$.

It follows, if we choose the constant c appropriately, that the measure $\mu^{(2)}_{\alpha,\lambda,2k+1}$ is absolutely continuous with respect to $\mu_{\alpha,\lambda,2k+1}$, that is, the

support of $\mu^{(2)}_{\alpha,\lambda,2k+1}$ is in $E_{\lambda,2k+1}(\alpha)$. This makes it natural to try to apply Theorem 1 to the measures $\mu^{(2)}_{\alpha,\lambda,2k+1}$. Moreover, it is not difficult to see that $\mu^{(2)}_{\alpha,\lambda,2k+1}$ converges weakly to μ_λ, the distribution of the corresponding Bernoulli convolution, as $k \to \infty$.

We will now prove the following result for the measures $\mu^{(2)}_{\alpha,\lambda,2k+1}$, analogous to Proposition 2.

Proposition 4. *Let $\alpha s < 1$. If $\lambda^2 \in (\frac{1}{2}, 0.64)$, then, almost surely, there exists a constant C such that*

$$\iint |x-y|^{-s}\,\mathrm{d}\mu^{(2)}_{\alpha,\lambda,2k+1}(x)\mathrm{d}\mu^{(2)}_{\alpha,\lambda,2k+1}(y) \le C,$$

holds for infinitely many k.

Proposition 4 implies together with Proposition 1 the statement of Theorem 2 for almost all λ such that $\lambda^2 \in (\frac{1}{2}, 0.64)$, that is for almost all $\lambda \in (\frac{1}{2}, \frac{4}{5})$.

Proof of Proposition 4. To prove Proposition 4, we do as in Section 3.2, and write

$$\iint |x-y|^{-s}\,\mathrm{d}\mu^{(2)}_{\alpha,\lambda,2k+1}(x)\mathrm{d}\mu^{(2)}_{\alpha,\lambda,2k+1}(y) = c_s \int |\hat{h}^{(2)}_{\alpha,\lambda,2k+1}(\xi)|^2 |\xi|^{s-1}\,\mathrm{d}\xi.$$

Now, we use the fact that $h^{(2)}_{\alpha,\lambda,2k+1} = h_{2\alpha,\lambda^2,k,c} * (h_{2\alpha,\lambda^2,k,c} \circ S_1^{-1})$, or equivalently $\hat{h}^{(2)}_{\alpha,\lambda,2k+1} = \hat{h}_{2\alpha,\lambda^2,k,c} \cdot \widehat{(h_{2\alpha,\lambda^2,k,c} \circ S_1^{-1})}$, together with the Cauchy–Bunyakovsky–Schwarz inequality to conclude that

$$\iint |x-y|^{-s}\,\mathrm{d}\mu^{(2)}_{\alpha,\lambda,2k+1}(x)\mathrm{d}\mu^{(2)}_{\alpha,\lambda,2k+1}(y) \le c_{s,\lambda} \int |\hat{h}_{2\alpha,\lambda^2,k}(\xi)|^4 |\xi|^{s-1}\,\mathrm{d}\xi,$$

where $c_{s,\lambda}$ is a constant that only depends on s and λ. By Proposition 3 we now get that

$$\iint |x-y|^{-s}\,\mathrm{d}\mu^{(2)}_{\alpha,\lambda,2k+1}(x)\mathrm{d}\mu^{(2)}_{\alpha,\lambda,2k+1}(y) \le c_{s,\lambda}(C4^{(\alpha s-1)k} + D).$$

This clearly implies Proposition 4. □

We can now consider higher powers of convolutions of scalings of the measures $\mu_{\alpha,\lambda,k}$. Similarly as was done in Proposition 4, we can conclude the statement of Theorem 2 for almost all $\lambda \in (\frac{1}{2}, \sqrt[m]{\frac{4}{5}})$, if we prove, instead of Proposition 3, an estimate on

$$\int |g_{\alpha,\lambda,k}(\xi)|^{2^{m+2}} |\xi|^{2^{m+1}s-1}\,\mathrm{d}\xi,$$

analogous to Proposition 3. This can be done for any m in a straight-forward way, similar to the proof of Proposition 3, but would be somewhat lengthy and cumbersome. We will therefore leave out the details, since the proof of Proposition 3 contains all the necessary ideas. Since $\sqrt[m]{4/5} \to 1$ as $m \to \infty$, this concludes the proof of Theorem 2.

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